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S.M. Nikol'skii's Works on the Theory of Approximation of Functions[1]

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Nikol'skii's early mathematical studies concerned the theory of linear operators in linear normed spaces. In this branch, he obtained profound results related to the validity of the Fredholm alternative for linear equations.

During the next years until 1951, Nikol'skii worked on various problems in the theory of approximation of functions. This survey covers some of his fundamental results obtained during that period. Unfortunately, many of his other important results are beyond the scope of this paper. In this connection, we note Korneichuk's large article devoted to the works by Nikol'skii in approximation theory and their development [1].

In his 1951 work [2], Nikol'skii established inequalities in different metrics for trigonometric polynomials and entire functions of exponential type. These results laid a foundation for his own investigations and the studies by his numerous followers on the embedding theorems for spaces of differentiable functions of several variables and on their applications to problems of mathematical physics.

In subsequent years, Nikol'skii's investigations primarily concerned these subjects. They are surveyed in [3].

1. UPPER BOUNDS OF APPROXIMATIONS BY FOURIER SUMS ON FUNCTION CLASSES

Lebesgue was the first who started the analysis of the rate of approximation of periodic functions by partial sums of Fourier series [4]. As a characteristic of the approximation properties of Fourier sums on a function class $\mathfrak{M}$, he considered the upper bound of the deviations on this class:

$$S_n(\mathfrak{M}) = \sup_{f\in\mathfrak{M}} \|f(x) - s_n(f,x)\|_C. \tag{1.1}$$

For the class $H(\omega)$ of functions whose modulus of continuity does not exceed a prescribed value $\omega(\delta)$, Lebesgue proved the order equality

$$S_n(H(\omega)) \sim \omega\left(\frac{1}{n}\right), \qquad n \to \infty.$$

The next step in the study of upper bounds (1.1) was taken by Kolmogorov [5]. He derived an asymptotic formula for quantities (1.1) in the case when $\mathfrak{M}$ is the class W_C^r, $r = 1, 2, \dots,$ of functions with absolutely continuous derivatives of order $r-1$ and with the rth derivatives satisfying

[1]This report was made on May 3, 2000, at the International Conference "Modern Problems of the Theory of Functions and Function Spaces" dedicated to the 95th birthday of Academician Sergei Mikhailovich Nikol'skii (Moscow).

the condition $|f^{(r)}(x)| \le 1$ in its domain:

$$S_n(W_C^r) = \frac{4}{\pi^2}\,\frac{\ln n}{n^r} + O\biggl(\frac{1}{n^r}\biggr), \qquad n \to \infty. \tag{1.2}$$

The first important continuation of Kolmogorov's result was a series of Nikol'skii's studies of 1940–1946 [6–12]. Along with bounds (1.1) for Fourier sums, Nikol'skii also considered analogous upper bounds for approximations on function classes by Fejér sums and interpolating polynomials.

Let us cite Nikol'skii's result on the approximation by Fourier sums of the classes $W^rH(\omega)$, $r = 1, 2, \dots$, of functions with the derivatives $f^{(r)} \in H(\omega)$, where ω is a concave modulus of continuity:

$$S_n(W^rH(\omega)) = \frac{2}{\pi^2}\,\frac{\ln n}{n^r}\int_0^{\pi/2} \omega\biggl(\frac{2t}{n}\biggr)\sin t\,dt + O\biggl(\frac{1}{n^r}\,\omega\biggl(\frac{1}{n}\biggr)\biggr), \qquad n \to \infty. \tag{1.3}$$

Note that Nikol'skii devised a new research technique for deriving (1.3). This was motivated by the fact that the extremum functions in (1.3) are of a much more complicated nature than those in Kolmogorov's bound (1.2).

After Kolmogorov's and Nikol'skii's investigations, the asymptotic behavior of quantities similar to (1.1) was examined by numerous authors in various formulations. This problem is sometimes called the Kolmogorov–Nikol'skii problem.

2. DUALITY THEOREMS

The determination of upper bounds of the form (1.1) for classes of functions representable as convolution is reduced to the approximation of the convolution kernel in the metric of the dual space.

Nikol'skii considered this problem in the general form [12].

Let B be a Banach space and B^* be its dual space.

Fix elements $x_1, \dots, x_n$ of B. Then, for the best approximations of an arbitrary element $x \in B$ by polynomials in $x_1, \dots, x_n$, we have

$$\min_{a_k} \biggl\| x - \sum_{k=1}^n a_k x_k \biggr\|_B = \max_F F(x),$$

where the maximum is taken over all linear functionals $F \in B^*$ such that $\|F\|_{B^*} \le 1$ and $F(x_1) = \ldots = F(x_n) = 0$.

Another duality theorem of Nikol'skii deals with the case when linear functionals $F_1, \dots, F_n$ from B^* are fixed and an arbitrary functional $F \in B^*$ is approximated by polynomials over the system $F_1, \dots, F_n$. Then,

$$\min_{a_k} \biggl\| F - \sum_{k=1}^n a_k F_k \biggr\|_{B^*} = \sup_x F(x),$$

where the supremum is taken over all elements $x \in B$ such that $\|x\|_B \le 1$ and $F_1(x) = \ldots = F_n(x) = 0$.

Nikol'skii's duality theorems served as a starting point for further investigations of extremum problems in functional analysis. Moreover, they opened approaches to solving many problems in the theory of approximation of functions.

3. APPROXIMATION IN THE MEAN

Nikol'skii was the first who applied the duality theorems to the problems of the theory of approximation [12].

Let W_L^r, $r = 1, 2, \dots$, be a class of 2π-periodic functions f with the rth derivatives satisfying $\|f^{(r)}\|_L \le 1$. Then, for the upper bounds of the best approximations of functions from W_C^r by trigonometric polynomials of degree n in the metric of C and for the upper bounds of the best approximations of functions from W_L^r in the metric of L, we have

$$E_n(W_C^r)_C = E_n(W_L^r)_L, \qquad n = 1, 2, \dots. \tag{3.1}$$

The values of $E_n(W_C^r)_C$ had been found by Favard by that time [13]. Thus, (3.1) gave the values of $E_n(W_L^r)_L$ as well.

An equality of the form (3.1) was also derived in [12] for the best approximations of functions from the classes $\overline{W}_C^r$ and $\overline{W}_L^r$ that are conjugate to functions from W_C^r and W_L^r, respectively.

Moreover, Nikol'skii established that an analogous equality holds for the classes of functions f representable as a convolution

$$f(x) = \frac{a_0}{2} + \frac{1}{\pi}\int_{-\pi}^{\pi} K(t - x)\varphi(t)\,dt,$$

where $K(t) \in L$ is the convolution kernel satisfying certain conditions and φ is a function such that $\|\varphi(x)\|_C \le 1$ or $\|\varphi(x)\|_L \le 1$.

For a wide class of kernels $K(t)$, Nikol'skii established necessary and sufficient conditions for the upper bounds of the best approximations of the corresponding function classes to coincide in the C and L metrics.

4. APPROXIMATION BY ALGEBRAIC POLYNOMIALS WITH AN IMPROVED DEGREE OF APPROXIMATION NEAR THE INTERVAL ENDPOINTS

To facilitate the comparison between the approximations of periodic functions and the approximation of functions by algebraic polynomials, the latter is usually examined on the interval $[-1, 1]$. For simplicity, we restrict our consideration to the classes $\operatorname{Lip}\alpha$ of functions f satisfying on $[-1, 1]$ the Lipschitz condition of order α:

$$|f(x') - f(x'')| \le M|x' - x''|^\alpha, \qquad M > 0, \quad 0 < \alpha \le 1. \tag{4.1}$$

In the first studies of 1911–1919 devoted to direct and inverse theorems of approximation theory, D. Jackson, S.N. Bernstein, and C.J. de la Vallée Poussin established that the direct theorems are analogous in the trigonometric and algebraic cases: if $f \in \operatorname{Lip}\alpha$, then

$$E_n(f)_C \le \frac{A}{n^\alpha}, \qquad n = 1, 2, \dots, \tag{4.2}$$

where A is an absolute positive constant.

However, the inverse theorems differ significantly in the trigonometric and algebraic cases. Specifically, if (4.2) with $\alpha < 1$ holds for the best approximation of a periodic function f, then f satisfies the Lipschitz condition (4.1) of order α with a constant M.

For approximations by algebraic polynomials, estimate (4.2) with $\alpha < 1$ does not imply that f belongs to $\operatorname{Lip}\alpha$ on the entire $[-1, 1]$. One can only state that f satisfies the Lipschitz condition of order α on every interval $[a, b] \subset (-1, 1)$, and the factor M in (4.1) depends on the choice of $[a, b]$.

Thus, the characterization of functions satisfying the Lipschitz condition on an interval in terms of their approximations by algebraic polynomials remained an open problem.

An approach to solving this problem was developed in Nikol'skii's work of 1946 [14].

For a given function $f \in \mathrm{Lip}\,1$ on $[-1,1]$, he constructed in [14] a sequence of algebraic polynomials $P_n(x)$ of degree at most n such that

$$|f(x) - P_n(x)| \le \frac{\pi}{2}\frac{\sqrt{1-x^2}}{n} + O\biggl(|x|\frac{\ln(n+1)}{n^2}\biggr) \tag{4.3}$$

uniformly with respect to $x \in [-1,1]$. For these polynomials, O in (4.3) at $x = \pm 1$ cannot be replaced by o, and, more importantly, the factor $\pi/2$ cannot be diminished, which follows from Nikol'skii's another result stated in [14]:

$$\sup_{f \in \mathrm{Lip}\,1} E_n(f)_C = \frac{\pi}{2n} + o\biggl(\frac{1}{n}\biggr), \tag{4.4}$$

where the remainder is negative.

Estimate (4.3) improves the degree of approximation near the endpoints of $[-1,1]$. This circumstance led Nikol'skii to the following conjecture. For a function f to satisfy the Lipschitz condition of order $\alpha < 1$ on $[-1,1]$, it is necessary and sufficient that there exist a sequence of algebraic polynomials $\{p_n(x)\}$ such that

$$|f(x) - p_n(x)| \le A\biggl(\frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\biggr)^{\alpha}.$$

This conjecture was proved by Nikol'skii's students. A.F. Timan proved the corresponding direct theorem in 1951, and V.K. Dzyadyk proved the converse theorem in 1956.

Further investigations have showed that many theorems on the approximation of periodic functions by trigonometric polynomials also hold for approximations by algebraic polynomials on $[-1,1]$ if $1/n$ in the periodic case is replaced, as in Nikol'skii's conjecture, by

$$\frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}.$$

5. QUADRATURE FORMULAS

The study of quadrature formulas, i.e., approximate integration formulas of the form

$$\int_0^1 f(x)\,dx \approx \sum_{k=1}^{n} a_k f(x_k),$$

was initiated in the 17th century. A large variety of results on this subject have been stated since then.

In 1950, Nikol'skii [15] considered a new extremum problem for quadrature formulas: Given a function class $\mathfrak{M}$ and the number n of nodes in a quadrature formula, find

$$\inf_{a_k,\,x_k} \sup_{f \in \mathfrak{M}} \biggl|\int_0^1 f(x)\,dx - \sum_{k=1}^{n} a_k f(x_k)\biggr|, \tag{5.1}$$

where the infimum is taken over all weights a_k and all nodes x_k of the quadrature formula.

Nikol'skii examined general questions related to this statement of the problem and found its exact solution in certain cases.

In particular, he established that the determination of (5.1) in the class of functions $f(x)$ having a bounded second-order derivative and satisfying the initial conditions $f(0) = f'(0) = 0$ requires finding the best approximation, in the $L[0,1]$ metric, of the function x^2 by continuous piecewise linear functions whose derivative may have n discontinuity points. In modern terminology, the problem is to approximate x^2 by splines of degree 1 of defect 1 with n free nodes.

Nikol'skii proved that, for this class of functions, the best nodes and weights in problem (5.1) are

$$x_k = k\alpha_n, \quad k = 1, 2, \ldots, n, \qquad \alpha_n = \frac{4}{\sqrt{3} + 4n},$$
$$a_k = \alpha_n, \quad k = 1, 2, \ldots, n-1, \qquad a_n = \alpha_n \frac{2 + \sqrt{3}}{4}.$$

Thus, the nodes of the optimal quadrature formula are uniformly distributed over $[0,1]$, except the last (nearest to 1) node, which is closer to the interval endpoint, while the weights of all nodes, except the last, are equal.

In those years, the study of spline approximation took its first steps, and this result of Nikol'skii was among the first ones that gave an exact solution to a spline approximation problem.

Nikol'skii also obtained similar results for quadrature formulas involving not only the function itself but also its derivatives.

In 1958, Nikol'skii's monograph *Quadrature Formulas* [16] was published, which was later republished several times. This book gave impetus to further investigations of extremum problems for quadrature formulas and promoted the application of theoretical results in computational practice.

6. INEQUALITIES IN DIFFERENT METRICS FOR TRIGONOMETRIC POLYNOMIALS

In [2], Nikol'skii obtained estimates relating the norms of trigonometric polynomials and the norms of entire functions of exponential type in L^p metrics with different p. For simplicity, we consider this problem only for trigonometric polynomials.

The exact statement of the problem is as follows. Let $T_{n_1,\ldots,n_m}(x_1, \ldots, x_m)$ be trigonometric polynomials in m variables such that their degrees in each of the variables x_k is at most n_k, $k = 1, \ldots, m$. Given p and q such that $1 \le p < q \le \infty$, estimate the norm of the polynomial $T_{n_1,\ldots,n_m}(x_1, \ldots, x_m)$ in L^p in terms of its norm in L^q. The condition $p < q$ is natural since the corresponding norms, regarded as functions of $n_1, \ldots, n_m$, may have the same order when $p > q$.

Nikol'skii proved the estimate

$$\|T_{n_1,\ldots,n_m}\|_{L^p} \le A_m (n_1 \ldots n_m)^{\frac{1}{p} - \frac{1}{q}} \|T_{n_1,\ldots,n_m}\|_{L^q}, \tag{6.1}$$

where A_m depends only on m. Here, the order of the quantity

$$(n_1 \ldots n_m)^{\frac{1}{p} - \frac{1}{q}}$$

cannot be reduced. Earlier, such an estimate was known only for $m = 1$, $p = 2$, and $q = \infty$ [17].

In addition to (6.1), Nikol'skii obtained in [2] an estimate for different numbers of dimensions:

$$\|T_{n_1,\ldots,n_m}\|_{L^p(x_1,\ldots,x_k)} \le A_m (n_{k+1} \ldots n_m)^{\frac{1}{p}} \|T_{n_1,\ldots,n_m}\|_{L^p(x_1,\ldots,x_m)}, \qquad 1 \le k < m,$$

where $1 \le p \le \infty$ and A_m depends only on m. Here, the order of

$$(n_{k+1} \ldots n_m)^{\frac{1}{p}}$$

cannot be reduced either.

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